

XXIV. *Directions for making a Machine for finding the Roots of Equations universally, with the Manner of using it : By the Rev. Mr. Rowning, to John Bevis, M. D. F. R. S.*

S I R,

Read May 3, 1770. **P**ERUSING a discourse in the memoirs of the Royal Academy at Peterburgh, Tome vii. page 211, by the learned John Andrew de Segner, containing an universal method of discovering the roots of equations, which you was so kind as to recommend to my consideration; I found, that the author's method, as you observed to me, consisted in finding several ordinates of a parabolic curve, such, that its abscissas being taken equal to any assumed values of the unknown quantity in the equation, the ordinates corresponding to those abscissas, should be equal to the values of all the terms in the equation (when brought to one side) that is, in other words, in finding several ordinates of a parabolic curve defined by the equation proposed: in which case, as is well known, if a curve be drawn through the extremities of the said ordinates, the points upon the axis, where the curve shall cut it, will necessarily give the several values of the *real* roots

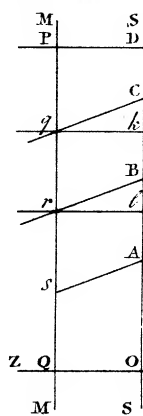
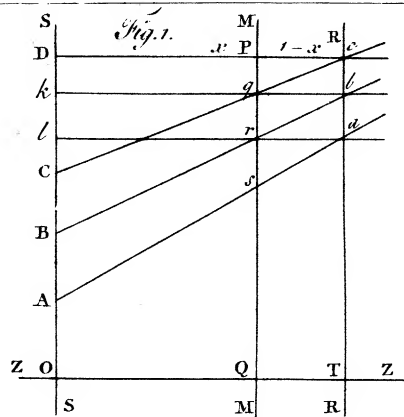
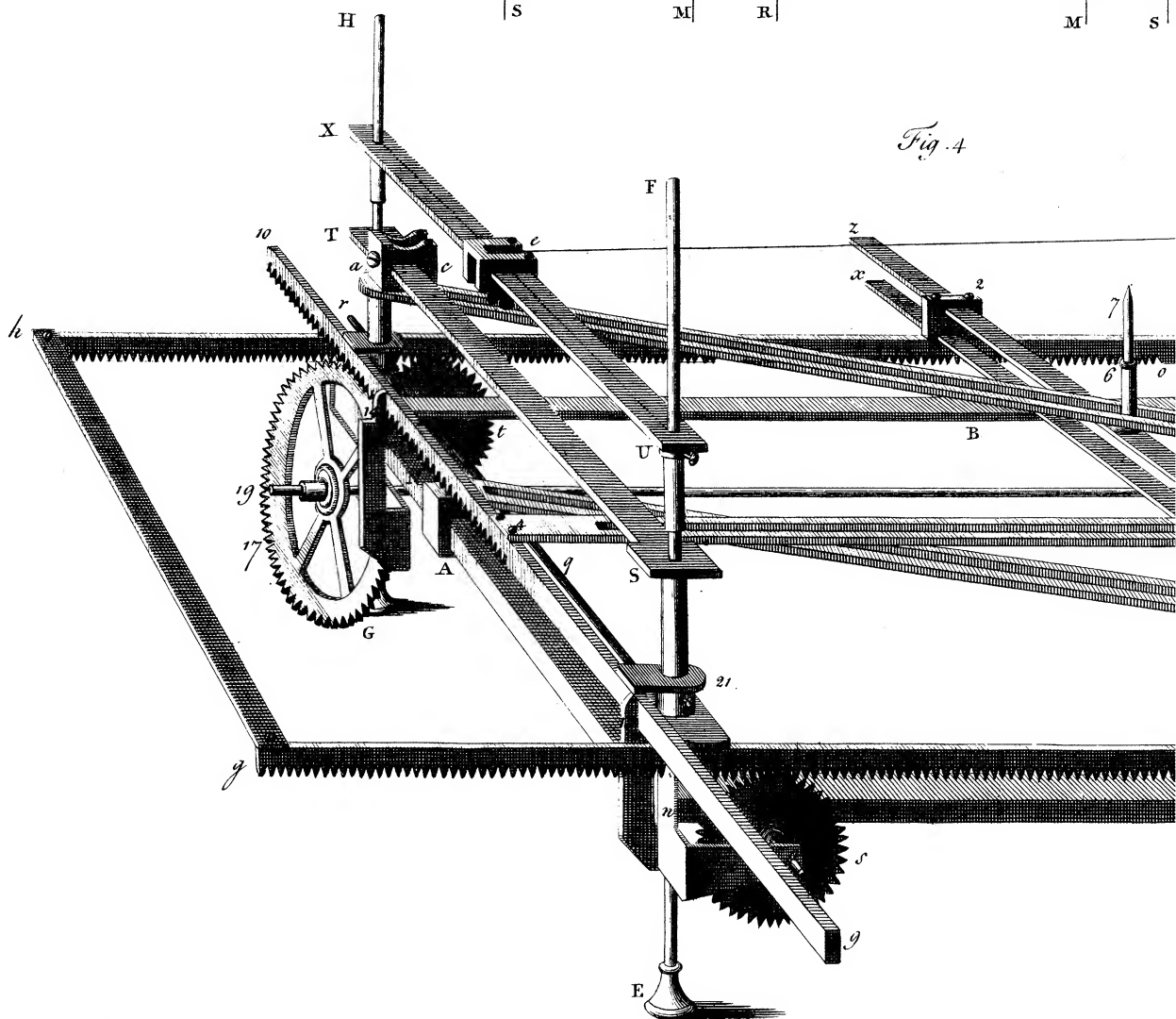
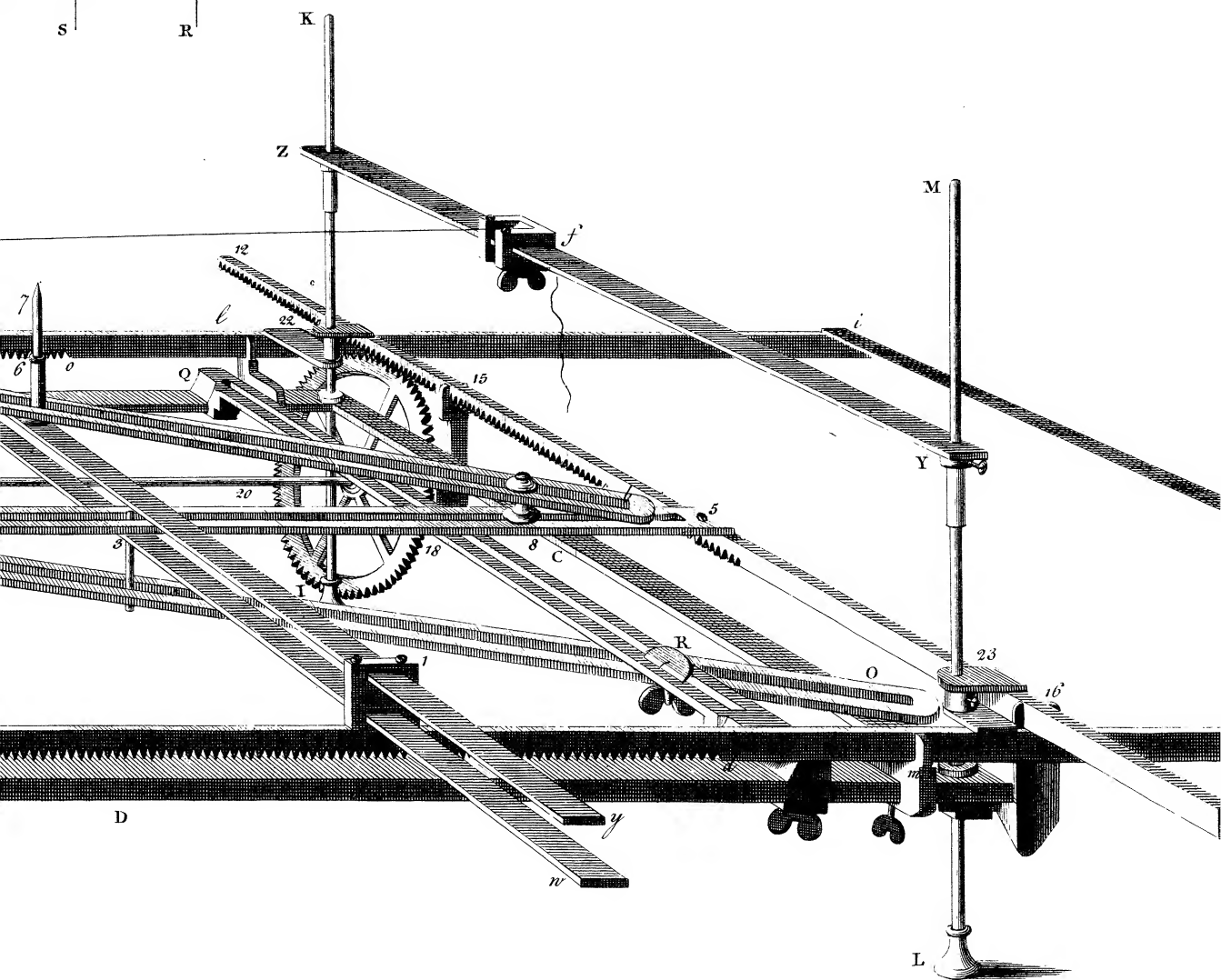
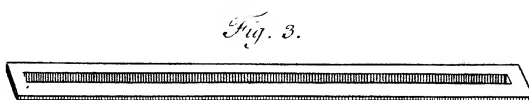
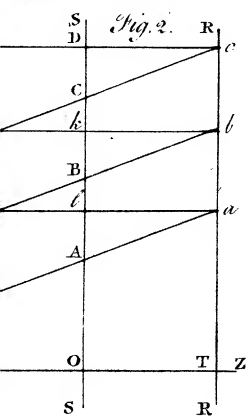
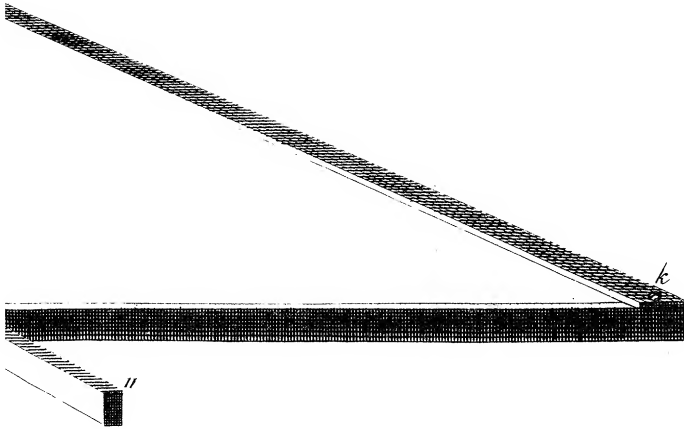


Fig. 4.







roots of the equation ; and the several points, where the curve shall approach the base, but shall return without reaching it, will shew the *impossible* ones.

This is a method I myself fell into ten or twelve years ago, and have constantly used for finding the roots of such equations as I have had occasion to consider. But his method is preferable to mine in one respect, *viz.* that whereas I always compute the value of the ordinates in numbers, he finds them by drawing certain right lines ; however, when there are both possible and impossible roots in an equation, as generally there are, these methods are both of them extremely embarrassing : the learned author therefore wishes, that some method might be thought of, whereby such curves, as we are now speaking of, might in all cases be described by *local* motion ; but this, he tells us, he looked upon as so very difficult a task, that he never attempted it. *Quod ad descriptionem attinet*, says he, *motum excogitare, quo tales accuratè designari possunt omnes [hujusmodi curvæ] admodum difficile judico, quare id neque tentavi.* This hint, however, convinced me, that the thing was possible ; I therefore determined to endeavour it.

I soon found, that if rulers were properly centered, and so combined together, that they should always continue representatives of the several right lines, by which he discovers the abovementioned ordinates, upon moving the first, a point or pencil, so fixed as to be carried along perpetually by the intersection of the first and last rulers, would describe the required curve, let the number of dimensions in the equation be what it will ; only the greater that number, the greater must be the number of the rulers made use

of. And this appeared to me so obvious, that I wondered, neither the learned author, who seems to have the thing much at heart, nor any body else since the publication thereof, saw it.

Let the equation to be resolved, be $a + bx + cxx + dxxx$, &c. = 0.

Upon the line ZZ , as a base, in either the first or second figure, draw the lines SS and RR perpendicular thereto, at any distance taken at pleasure from each other. Then upon the line SS in either figure, TAB. VIII. set off the lines OA , AB , BC , CD , &c. proportionable to the coefficients a , b , c , d , &c. in the equation, remembering to take each line upwards from the extremity of the last, when the coefficient, it is to represent, is *affirmative*; but downwards from the said extremity, when the coefficient is *negative*. Then through the extremity of the last of the lines OA , AB , BC , &c. which in our case is D , draw the line Dc , parallel to the base ZZ , and through c , where Dc intersects RR , draw cC ; and parallel to SS , and at any distance from it taken at pleasure, draw MM : then parallel to Dc , and where Cc intersects MM , draw the line kb ; through b , where this last line intersects RR , draw bB ; parallel also to Dc , and where bB intersects MM , draw la ; and through a , where la intersects RR , draw aA . This done, let SS , RR and Cc , be supposed to represent three rulers with grooves or notches cut quite through them, of such form as is represented Fig. 3. and let those rulers be screwed down in their respective places SS , RR , and Cc , to a plane or frame of sufficient size.

Then

Then let other rulers of like form, as Bb , Aa , &c. be *moveable* upon the centers B , A , &c. and let those centers themselves be moveable upwards and downwards in the groove or notch in the ruler SS , and in such manner, that the centers B and A may be placed upon one another, or upon C , if occasion requires, and let those centers be screwed down in their proper places, *viz.* the center A at A , the center B at B , &c. Then let kb and la represent other moveable rulers, such as the former, and so confined that they shall always continue parallel to themselves and to the line Dc ; and let MM represent another ruler of like form. And let the rulers kb and MM be connected with the fixed ruler Cc by a sliding pin passing through the intersection of their notches at q , and let the rulers kb , Bb , la and Aa be connected with one another and with MM and RR , by like pins passing through their notches at b , r , a , and s ; and let the last of these pins s , have the point of a pencil fixed therein; then, I say, that if the ruler MM be moved backwards or forwards, from or towards SS , and always held parallel thereto (or be so confined as to be capable of moving in no other manner) the pencil s shall describe the required curve. And the distances from the point O , at which that pencil shall cross the base ZZ , on the right hand side of SS , shall denote the *affirmative* roots of the equation; and the distances from the same point O , where it shall cross the said base on the left hand side of SS , shall express the *negative* ones; and the places, where it shall approach the said base, but shall return without reaching it, shall indicate the *impossible* ones; the said distances being

always to be estimated on a scale on which the assumed line Dc is *unity*.

Demonst. Since the lines OA , AB , BC , &c. are to be taken *proportional* to the coefficients a , b , c , &c. let us suppose the first of them, *viz.* OA , to be taken equal to the first coefficient a , or to any part of it taken at pleasure; suppose for instance, to the n^{th} part, that is, to $\frac{a}{n}$; then, to preserve the above-mentioned proportionality, the next *viz.* AB will be equal to $\frac{b}{n}$, BC will be equal to $\frac{c}{n}$, and CD to $\frac{d}{n}$, &c. Call OQ or its equal DP , x ; then Dc , as above-mentioned, being taken equal to *unity*, Pc will be equal to $1-x$, and DC being equal to $\frac{d}{n}$, and the triangles DCc and Pqc being similar, we have this proportion, *viz.* $1 : 1-x :: \frac{d}{n} : \frac{d-dx}{n} = Pq$ or Dk ; but kB is equal $BC+CD-Dk$, that is, to $\frac{c}{n} + \frac{d}{n} - \frac{d-dx}{n}$, that is, to $\frac{c+dx}{n}$; and by similar triangles, as $kb : qb :: kB : qr$, that is, in symbols, as $1 : 1-x :: \frac{c+dx}{n} : \frac{c+dx-cx-dxx}{n} = qr$ or kl : but Al is equal to $AD-Dk-kl$, that is, in symbols, to $\frac{b}{n} + \frac{c}{n} + \frac{d}{n} - \frac{d-dx}{n} - \frac{c+dx-cx-dxx}{n}$, that is, to $\frac{b+cx+dxx}{n}$; and by similar triangles, $la : ra :: Al : rs$; in symbols $1 : 1-x :: \frac{b+cx+dxx}{n} : \frac{b+cx+dxx-bx-cxx-dxxx}{n} = rs$; Qs therefore, which by the figure is equal to

QP

$$QP - Pq - qr - rs, \text{ is equal to } \frac{a+b+c+d-d-dx}{n} \\ - \frac{c+dx-cx-dxx}{n} - \frac{b+cx+dx-dx-bx-cxx-dxxx}{n},$$

that is, to $\frac{a+bx+cx+dx}{n}$; consequently, when Q_s is nothing, that is, when the curve described by s , cuts the base, $\frac{a+bx+cx+dx}{n}$ will be equal to nothing, and therefore equal also to $a+bx+cx+dx$; this last being also, from the equation proposed, equal to nothing; Q_s therefore in those circumstances will be equal to $a+bx+cx+dx$; and consequently whatever value of x or OQ renders $a+bx+cx+dx$ equal to nothing, will render Q_s equal to nothing: but every value of x that renders $a+bx+cx+dx$ equal to nothing, is a root of the proposed equation $a+bx+cx+dx=0$; consequently the curve will cross the base ZZ at every *real* root of that equation, whether negative or affirmative, and therefore, as every one acquainted with curve lines knows, will attempt to do so, but not quite reach it, at every *impossible* one. Q, E, D.

This demonstration is adapted only to an equation of three dimensions; but it is easy to see, it may be extended to any other.

Note. To obtain the negative roots, the rulers must be extended to the left of the line SS , as represented fig. 2. where they are denoted by the same letters as in the other figure; *viz.* the ruler Cc must be extended from c to q ; the ruler Bb from b to r , and aA from a to s , and onwards towards the left, the two last turning upon the centers A, B , in the fixed line SS .

It

It is not necessary, that the curve should be described with accuracy, or even that it should fall upon the plane or frame, except where it crosses or attempts to cross the base; and therefore there will arise no inconveniency in this respect from taking the lines OA, AB, &c. large. But the immoveable rulers OD and Tc must be fixed so near together, that, their distance Dc or OT, being reckoned *unity*, the base OT extended to the extremity of the plane to the right, may take in all the *affirmative* roots: and when extended to the left, may take in all the *negative* ones.

There is one thing more, very well worth observing, *viz.* that if you have an equation of this sort $xxx - 5xx + 1200x + 9000 = 0$ (where you may observe, that the coefficients unity, 5, 1200 and 9000 are so very different from each other, that it would be difficult to set them off in due proportion upon the line OD) you may reduce them into a more manageable form, by the following contrivance: instead of each x in the equation put $10x$, $20x$ or $100x$, suppose $20x$; then instead of xxx you will have $8000xxx$, instead of $-5xx$ you will have $-2000xx$, &c. and the whole equation will stand thus, $8000xxx - 2000xx + 24000x + 9000 = 0$; then divide each term by 100, and you bring it to $8xxx - 2xx + 24x + 9 = 0$, a much more tractable equation than the other: but then it must be remembered, that as you made x 20 times less than it was, the roots when discovered will be so many times less than the true ones, and therefore each must be multiplied by that number 20.

The following observations relative to the application of the abovementioned rulers may be useful:

1. The roots of equation are of three sorts, *affirmative*, *negative*, and *impossible*; which last are sometimes called *imaginary*.

2. Every equation contains as many roots as it has dimensions.

3. The impossible roots go by pairs. Thus, if an equation has an impossible root of this form $a + b\sqrt{-1}$, it has another impossible one of this, *viz.* $a - b\sqrt{-1}$, which may be called its fellow: from hence it follows, that every equation, if it has any impossible roots, has either *two*, *four*, or *six*, &c. that is, an even number of them: and every time the curve described by the rulers abovementioned approaches the base, and leaves it again without crossing it, it implies one impossible root and its fellow; so that if it approaches the base in this manner three times, it implies, that the equation has six impossible roots. And this is all the rulers can do with regard to this sort of roots; they cannot shew *what* those are, but only *how many* there are. I shall direct to a method of finding what they are in the 8th article below. Inasmuch then, as the impossible roots go by pairs, and the number of roots in any equation is equal to the number of its dimensions, it follows,

4. That every equation of an odd number of dimensions, must contain at least one *real* root.

5. Every equation whose first and last terms (when brought to one side) have contrary signs, will have at least also one real root: and therefore, when this is the case, and the number of its dimensions is

also *even*, it must have two real roots at least : because the number of dimensions being *even*, and the number of impossible roots always *even*, the number of real ones must be *even* also.

6. If any equation be divided by the unknown quantity *minus* one of its roots, it will be reduced one dimension lower. And as every equation contains as many roots as it has dimensions, it follows,

7. That if you deduct the number of impossible roots from the whole number of its roots, that is, from the number of its dimensions, the remainder will give the number of its real roots.

8. When you have found by the rulers, what those real roots are, put the unknown quantity (x) equal to each of them, transpose the terms in each equation to one side, multiply all the equations together, and divide the equation proposed by their product ; then make the quotient equal to nothing, and you have an equation containing all your impossible roots, without any real ones intermixed. Then those impossible roots may be found by the method for that purpose laid down by Monf. de *Bougainville* in his *Traité du Calcul integral*, in the fifth and sixth chapters of his introduction ; and which is the best method I know of.

His method consists in parting the equation into two others, of the same number of dimensions indeed, but such as shall involve no other than real roots ; which real roots you may then find by these rulers, or otherwise ; and from thence you will obtain all the impossible roots of your equation. But because few English mathematicians, I suspect, are acquainted
with

with this method, it may be useful to give the substance of it here in our own language.

The author previously demonstrates the two following proportions.

PROP. I. That when a quantity is equal to nothing, and is composed of many terms, some of which are *real*, and the other are terms *multiplied by* $\sqrt{-1}$, the sum of the real ones is equal to nothing; and the sum of those that are multiplied by $\sqrt{-1}$ is also equal to nothing. This is the 69th article of his introduction.

PROP. II. That when an equation involves imaginary roots only, the unknown quantity may always be supposed equal to $m+n\sqrt{-1}$; m and n being real quantities. This is the 80th article of his introduction.

Then to find the roots of such an equation as we are speaking of, for every unknown quantity (x suppose) in the equation, substitute $m+n\sqrt{-1}$, and you will obtain a new equation involving real terms, and terms multiplied by $\sqrt{-1}$; the former of which by Prop. I. are always equal to nothing, and so are the latter: make them so therefore, and you have two equations, from which the two assumed quantities m and n may be discovered; and consequently, since the value of x is by the second proposition equal to $m+n\sqrt{-1}$, it is discovered also.

What I mean in the former part of this article may be explained by the following instance; suppose the real roots discovered by the rulers abovementioned to be $a, b, -c$, &c. then put $x=a, x=b, x=-c$, &c. transpose the terms to one side, and you have $x-a=0, x-b=0, x+c=0$, &c. multiply all these last

equations together, divide the equation proposed by their product, and proceed as abovementioned.

9. The greatest negative coefficient of any equation (considered as affirmative) and increased by unity always exceeds the greatest affirmative root of the equation. And therefore,

10. If for the unknown quantity (x) in the equation, you put that coefficient taken affirmatively and increased by unity minus x , all the roots of the equation will be rendered affirmative. If you do this, you need only use such rulers, as are described in the first figure, whose centers are at their extremities, and so one sort of rulers will be sufficient for all cases. For you may observe, those in the second figure are different from the other, as to their centers.

11. If, when you have made all the roots of your equation affirmative, you would avoid removing the ruler MM to the right hand side of RR , which might be attended with inconveniency; that is, if you would have all the roots of your equation fall between O and T , that is, between nothing and unity, instead of the unknown quantity x in your last equation, put x multiplied by the greatest negative coefficient therein considered as affirmative and increased by unity; for instance, if the greatest negative coefficient in the equation be minus 9, put $10x$ instead of every x in the equation, and you will have a new equation, all whose roots shall fall upon the line OT unproduced; for then they will be less than unity, that is, than DC or OT : but when the roots are thus found, each of them must be multiplied by that coefficient increased by unity, that is, in the above instance, by 10; because putting $10x$
for

for x , is making every root ten times smaller than it is.

These propositions are all taken from the writings of the algebraists, or however are such as easily follow from them, and therefore need not be demonstrated here.

The following is a description of a machine for regulating the motion of the abovementioned rulers, which I caused to be made by an excellent workman in this town, and which I desire the society to accept of, to be kept as a specimen, for the inspection of any gentleman, who may chuse to have such made. It extends only to equations of two dimensions; but it is easy to see from it, how it may be carried to others of any number. A draft of it is exhibited in fig. 4. of the abovementioned table: where ABCD represents a frame of iron or steel, consisting of four strait bars joined together at their extremities, and forming a rectangular parallelogram about 12 inches long and 8 broad; into which at its four corners are screwed four perpendicular columns EF, GH, IK and LM, whose lower ends serve it as feet to stand on. And on one of the aforesaid bars, viz. on A, is a moveable nut or slider, which may be screwed to it at any point thereof; it appears in the figure at N; and on this nut, as a center, one end of the bar NO turns, whose other end is screwed down to the cross bar PQ at R, which cross bar is screwed down to the frame at P and Q, and may be set nearer or further from the end A at pleasure: this bar represents the line RR in fig. 1. Then

on the perpendicular pillars EF, GH, IK and LM, are fixed three bars ST, UX and YZ; on the first of these ST, is a sliding nut *c*, which carries one extremity of the bar *ab* turning upon it as a center; on the second and third, *viz.* UX and YZ, are also two nuts *e* and *f*, which may be screwed on any part of them, and on which the filken cord *ef* is fastned. The two first of those bars, *viz.* ST and UX together with the bar A, or rather lines such as the pricked line upon the upper one drawn upon them from pillar to pillar, represent the line SS in fig. 1. and the filken cord extended from the nut *e* to *f* and fixed to them, represents the base line ZZ in that figure.

Then besides these there is another rectangular figure *gbik*, about twice the length of the former, whose sides, *gk* and *bi* slide in grooves or supporters screwed to the frame ABCD, at proper places (three of which may be seen at *l*, *m*, and *n*), and have triangular teeth on their under sides from *g* to *d*, and from *b* to *o*, which run in similar teeth in two wheels *s* and *t*, of equal diameters, fixed on an axis *pqr*, and their axis *pr* is supported by proper cocks, one of which appears at *u*, the other not seen in this view of the figure. These wheels being both fixed to the same axis, and the triangular teeth in the bars fitting those in the wheels exactly, occasion the bars *gk* and *bi* to move with equal paces, when the machine is put in motion; by which means the bars *wx* and *yz*, which slide in two pieces 1 and 2 screwed to the said bars *gk* and *bi*, necessarily move parallel to themselves: these bars represent MM in fig.

fig. 1. In the lower of these bars wx is fixed a perpendicular pin, at 3, whose upper end passes through the groove of the bar 4, 5, and its lower end through that of the bar NO , and in the upper of these bars yz , is fixed a perpendicular pin 6, 7, the upper end of which may be taken out, and a pencil put in its place; this pin represents the point s , and the former pin 3, the point r , in fig. 1. There is also a perpendicular pin screwed to the bar 4, 5, at 8, which must be fixed upon that bar directly over the groove in the bar PQ . This pin denotes the point a in fig. 1. There are also two bars 9, 10, and 11, 12, with triangular teeth, which bars slide in supporters screwed to the frame at proper places (which appear at 13, 14, 15, and 16), and ride upon the wheels 17 and 18, with similar teeth, and both fixed on the axis 19, 20. These wheels prevent the said bars from moving with unequal paces, and therefore cause the bar 4, 5, which is screwed down to them at each end of it, to move parallel to itself; this bar represents la in fig. 1. Then the aforefaid nuts or sliders e , f , c , N , and R , being screwed down at proper places according to the coefficients of the equation (as shall be directed more particularly in the next article), and the bar gh being pushed forwards or backwards by the hand, will put the whole in motion; and the pin 6, 7, will describe a curve, which shall be the *locus* of the equation, and the distances where it shall pass under the filken cord ef , reckoning from the pricked line upon the bar UX , shall denote its real roots; and as many times as that pin shall approach the said cord,
and

and then recede from it without passing under it, twice so many will be the number of impossible roots in that equation. N. B. There are some small pieces screwed on to the perpendicular pillars EF, GH, IK and LM, three of which appear at 21, 22, and 23; these serve only to prevent the bars which slide under them from rising up.

To rectify the machine for a proposed equation.

Screw the nuts *e* and *f*, which carry the silken cord at any equal distances from the pillars EF and LM. Then slide the nut *c*, which carries the extremity of the bar *ab*, till it is farther from the pillar EF than the place you have fixed the nut *e* at, by a number of divisions upon any scale of equal parts, equal to the known term of the equation, if that term be affirmative; but so much nearer to it, if that term be negative; and fix it there. Then slide the nut N that carries the bar NO, till it is farther from or nearer to the pillar EF, than the last nut *c* is, by a number of divisions taken from the same scale, equal to the second coefficient of the equation (I mean that, where the unknown quantity is of one dimension); farther from it, if that coefficient is affirmative; but nearer, if it is negative: and fix it there. Then, lastly, slide the nut R, which fixes the other end of the bar NO, till it is farther from a line drawn from the pillar EF to LM, that is, farther from the side D of the frame than the last nut N is from the same, as many divisions, as the coefficient of that term of the equation, where the unknown quantity is of two dimensions, indicates; farther,

ther, if that coefficient is affirmative, otherwise nearer to it: for which purpose the end A of the frame, and the bars ST, UX, YZ and the cross bar PQ should all be graduated, beginning at the front of the frame D. They are graduated on the machine itself in a manner somewhat different, but it is found not so convenient for use. Then the several distances upon the cord *ef*, where the pencil or pin 6, 7, shall cross it, being reckoned from the prick'd line upon the upper bar UX, and measured upon a scale, on which the distance of the cross bar PQ from a line drawn along the middle of the end A, from EF to GH, is *unity* (the reason of which appears from the foregoing demonstration, where Dc or OT in fig. 1. and which answers to the distance of this line PQ from the bar A, was put equal to *unity*) shall give the required roots. And if the cord *ef* is removed, and a piece of pasteboard put over the machine, on the two upper bars UX and YZ, having a strait line drawn on its under side, representing the cord *ef*, and a pencil with its point upwards be put in the place of the pin 7, that pencil will describe on the under side of the pasteboard a curve, which with the said right line shall *construct* the equation proposed: and the larger the coefficients of the equation are (which coefficients may be made as large as you please without altering the roots, by multiplying them by any number at pleasure) the larger will be the angles at which the curve and the strait line shall cut each other; which in the construction of equations is a very desirable circumstance. And as it appears from the foregoing demon-

demonstration, that this machine by the application of more bars, may be extended to equations of *all dimensions*, it may not, I think, be improperly called an *Universal constructor of Equations*. I am

S I R,

With great respect,

your most humble and

most obedient servant,

London, Essex-street,
March 24, 1768.

J. Rowning.

